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ORTHOTROPIC AND TRANSVERSELY ISOTROPIC STRESS–STRAIN RELATIONS WITH BUILT-IN COORDINATE TRANSFORMATIONS

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Abstract—Computationally convenient elastic stress–strain relations are given for orthotropic and transversely isotropic materials in terms of unit vectors that are aligned in the preferred material directions. Methods are proposed for determining the material symmetry planes from mechanical and thermal response. The restrictions on the elastic constants that are discussed by Jones (1975) are specialized to the case of transverse isotropy. It is shown that the laminated composite structures with high in-plane stiffness may easily violate these restrictions. The incompressible version of the stress–strain relations are presented for materials that do not deform under hydrostatic loading. This version applies to composite structures made with incompressible epoxy matrices and stiff fibers. For incompressible laminated structures the restrictions on the elastic constants are violated when the in-plane modulus E_T and the out-of-plane modulus E_A are governed by $E_T > 4E_A$. Violation of the restrictions may be the cause of the common occurrence of interlaminar shear failure and edge delamination. In order to avoid excessive load transfer, it may be beneficial to keep the fiber matrix Young's moduli in the range $E_f < 20E_m$.

1. INTRODUCTION

The traditional approach for the determination of the stress–strain response of a material begins with microstructural observations in order to detect the overall structural symmetry in the material. Subsequently, the elastic constants of the material are measured along the macroscopic symmetry planes. The number of constants that are needed to completely characterize the elastic properties of the material depends on the number of the observed symmetry planes. Material symmetry axes are known as the "preferred" directions, or sometimes as the "principal" directions. However, we will reserve the term "principal", for describing the material planes over which shear stresses vanish under specific stress states. When the elastic stress–strain response is known in the preferred directions, the off-axes response can be determined from coordinate transformations, see Lai *et al.* (1978) or Nye (1979).

An alternative method for describing the off-axes elastic response is to include unit vectors that are attached to the preferred material directions in the constitutive (stress–strain) relationship (Spencer, 1971; Boehler, 1979). In this case, the constitutive relationship still contains the elastic constants that are measured in the preferred directions; however, both the stress and the strain are measured in the off-axes directions, and the need for these quantities being expressed in the preferred directions is eliminated, which may result in computational convenience as well as speed. Stress and strain are second order tensorial quantities and the directional unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ are vectors which are first order tensors. The mathematical form of the relationship between stress, strain and the material fixed unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ can be derived from the representation theorems for second order tensors (Boehler, 1979; Spencer, 1971). These theorems were used by Sutcu and Krempl (1984) and Sutcu (1985) to arrive at transversely isotropic and orthotropic visco-plastic stress–strain relations.

An inverse problem arises when we do not know the symmetry planes and the degree of anisotropy in the test specimen. Theoretically we can measure up to 21 elastic constants from the test specimen, however the measured values do not yield direct information regarding the number of possible symmetry planes. Woven fiber composites that are reinforced in non-orthogonal directions in three dimensions offer a good example. In this case, micromechanical models computationally produce 21 overall effective elastic constants for the structure (Pagano and Tandon, 1988, 1990). But we do not know if these are all independent constants or simply the result of an arbitrary rotation from a much simpler state of anisotropy which requires a lesser number of constants. Sometimes, single crystal materials are modeled as orthotropic or transversely isotropic materials, although the actual mechanical response is more complicated. For simple microstructures or weave architectures the possible symmetry planes can be determined by intuition. However, a general method and a related computational algorithm are needed for more complex structures.

In this paper, we will approximate the material response by transverse isotropy and orthotropy and then check the error that results from each case. It is desirable to approximate a given material response with transverse isotropy (if possible), rather than with orthotropy. Because the former is characterized with only five elastic constants, whereas the latter requires nine constants. Elasticity solutions for transversely isotropic materials are accordingly simpler than the orthotropic case. However, transverse isotropy is more restrictive for accurately modeling the material response.

It is well known (Nye, 1979) that a homogeneous material can have three independent linear thermal expansion coefficients in three mutually orthogonal directions. Thus, raising the temperature of a test specimen and measuring the uniform thermal strains offers a convenient method for determining three mutually orthogonal material directions in which the material may be orthotropic. The three eigenvalues and the three eigenvectors of the thermal strain tensor (normalized with respect to the temperature) correspond to the three independent thermal expansion coefficients and the three possible preferred directions, respectively.

The second method for determining three mutually orthogonal directions is to measure the strain tensor under uniform hydrostatic pressure. The eigenvalues and the eigenvectors of the resulting strain tensor can be directly related to the preferred directions and to the elastic constants, if the material happens to be orthotropic or possesses a simpler class of anisotropy. Otherwise, these directions do not correspond to the preferred material directions. It is not clear how one can use the information gained from hydrostatic loading in a useful manner in the case of more complicated anisotropy. It is possible that the thermal expansion anisotropy is a simpler class than actual structural symmetry, therefore the method of hydrostatic pressure is more reliable. Alternative methods such as uniaxial loading are needed for incompressible elastic materials.

Furthermore, in this paper we show that the effective elastic constants of laminated composites may violate the restrictions on the elastic constants. These restrictions are given in Jones (1975) for orthotropic elastic constants and ensure that the resulting structure can deform as a continuum under general loading.

2. ORTHOTROPIC STRESS STRAIN RELATION IN TERMS OF MATERIAL FIXED UNIT VECTORS $\bar{M}$, $\bar{N}$, $\bar{L}$

An orthotropic body has three mutually perpendicular planes of material symmetry. The three mutually perpendicular directions, which are the intersections of the symmetry planes, are called the preferred directions.

In Fig. 1 we show a material region D_1 with a global coordinate system XYZ attached to it. The three mutually orthogonal preferred directions are inclined at an arbitrary orientation in space. The coordinate system xyz is attached to the preferred directions. The region D_2 in Fig. 1 is cut from region D_1 parallel to the preferred directions and mechanically tested in order to obtain the stress–strain relation in the preferred directions. Our aim is to obtain the stress–strain relation in the global direction XYZ without any coordinate transformation. For this purpose, we will use the unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ in order to "remember" the preferred directions in our constitutive relationship.

As shown in Fig. 1 the unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ are attached to the preferred direction x, y and z, respectively, and given by

$$\bar{M} = M_1 i + M_2 j + M_3 k \tag{1a}$$

$$\bar{N} = N_1 i + N_2 j + N_3 k \tag{1b}$$

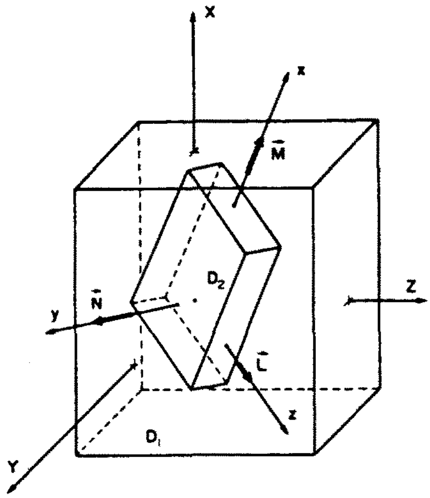

Fig. 1. The material fixed coordinate system xyz and the unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ are aligned along the preferred directions of the material. The material properties are determined from testing the subregion D_2 which is cut from region D_1 parallel to the orthotropy axes. Stress–strain relations are desired in the global directions XYZ for the region D_1. The material may be transversely isotropic in the yz plane.

$$\bar{L} = L_1 i + L_2 j + L_3 k, \tag{1c}$$

where the unit vectors i, j and k are in the global coordinate directions X, Y and Z, respectively, and the quantities M_1, M_2 and M_3 are the direction cosines of the preferred direction x. The quantities N_1, N_2, N_3, L_1, L_2 and L_3 are similarly defined.

The transformation of the strain tensor (or any second rank tensor) ε_{ij} is governed by

$$\varepsilon_{ij} = \varepsilon'_{11} M_i M_j + \varepsilon'_{22} N_i N_j + \varepsilon'_{33} L_i L_j + \varepsilon'_{23}(N_i L_j + L_i N_j) + \varepsilon'_{13}(M_i L_j + L_i M_j) + \varepsilon'_{12}(M_i N_j + N_i M_j), \tag{2}$$

where the indices i and j range from 1 to 3. The quantity ε'_{mn} refers to the components of the strain tensor in the preferred material directions xyz. The strain components in the global directions XYZ are denoted by ε_{ij} in (2). The inverse relationship to (2) is given by

$$\varepsilon'_{11} = M_p M_q \varepsilon_{pq}, \quad \varepsilon'_{22} = N_p N_q \varepsilon_{pq}, \quad \varepsilon'_{33} = L_p L_q \varepsilon_{pq}$$
$$\varepsilon'_{23} = N_p L_q \varepsilon_{pq}, \quad \varepsilon'_{13} = M_p L_q \varepsilon_{pq}, \quad \varepsilon'_{12} = M_p N_q \varepsilon_{pq}, \tag{3}$$

where summation over repeated indices is implied. The shear strain components ε_{23}, ε_{13} and ε_{12} in (2) and (3) are one half of their corresponding engineering shear strain components γ_{23}, γ_{13} and γ_{12} respectively. Equations (2) and (3) are useful for determining the preferred material directions from mechanical test data, as shown later.

2.1. *Preferred directions*

The stress–strain relations in the preferred directions xyz have the following form (Lekhnitskii, 1981)

$$\varepsilon_x = e_x - \alpha_x(T - T_i) = (\sigma_x - \nu_{xy}\sigma_y - \nu_{xz}\sigma_z)/E_x \tag{3a}$$

$$\varepsilon_y = e_y - \alpha_y(T - T_i) = (\sigma_y - \nu_{yx}\sigma_x - \nu_{yz}\sigma_z)/E_y \tag{3b}$$

$$\varepsilon_z = e_z - \alpha_z(T - T_i) = (\sigma_z - \nu_{zx}\sigma_x - \nu_{zy}\sigma_y)/E_z \tag{3c}$$

$$\sigma_{yz} = 2G_{yz}\varepsilon_{yz}, \quad \sigma_{xz} = 2G_{xz}\varepsilon_{xz}, \quad \sigma_{xy} = 2G_{xy}\varepsilon_{xy}. \tag{3d}$$

The total strains are denoted by e_{ij}, whereas ε_{ij} excludes thermal strains. The definition of the elastic constants E_x, E_y, E_z, G_{yz}, G_{xz}, G_{xy}, ν_{yz}, ν_{zy}, ν_{xz}, ν_{zx}, ν_{xy}, ν_{yx} can be obtained from Jones (1975), Lekhnitskii (1981) or more recently from Buczek (1986). The relationships between the Poisson's ratios and the restrictions on the values which the elastic constants can take are given in Jones (1975).

An orthotropic material has three independent linear thermal expansion coefficients α_x, α_y and α_z in the three principal directions. The rectangular prism D_2 in Fig. 1 which is aligned in the preferred directions remains rectangular upon temperature variation. However, the region D_1 in the global directions XYZ deforms in shear as well because of the off-axis thermal strains.

The relationships in (3a)–(3c) can be inverted to obtain the stresses σ_{ij} in terms of the mechanical strains ε_{ij}

$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z \tag{4a}$$

$$\sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z \tag{4b}$$

$$\sigma_z = C_{13}\varepsilon_x + C_{23}\varepsilon_y + C_{33}\varepsilon_z. \tag{4c}$$

The elastic constants C_{ij} are related to the engineering elastic constants E_x, E_y and so forth in Jones (1975).

2.2. *Incompressible orthotropic stress–strain relationship*

An elasticity theory was developed for fiber composites by Mulhern *et al.* (1967), Everstine and Pipkin (1973) and Spencer (1974) among others, in which the fibers were approximated by rigid inextensible cords and the matrix was assumed to be incompressible. For this reason, it is worthwhile to give the incompressible versions of the orthotropic stress–strain relationships in (3a)–(4c). It can be shown that the trace of the strain tensor ε_{ij} vanishes identically when the Poisson's ratios are given by

$$2\nu_{zy} = 1 + E_z/E_y - E_z/E_x \tag{5a}$$

$$2\nu_{zx} = 1 + E_z/E_x - E_z/E_y \tag{5b}$$

$$2\nu_{yx} = 1 + E_y/E_x - E_y/E_z. \tag{5c}$$

Thus, an incompressible elastic material possesses six constants, rather than nine. The resulting stress–strain law is again governed by (3a)–(3c) with the Poisson's ratios calculated from (5a)–(5c).

For the incompressible deformations, the inverse of (3a)–(3c) does not exist, because the hydrostatic stress cannot cause deformation. In physical terms, this may be interpreted as an indication that the material prefers failing by fracture under hydrostatic loading rather than deforming as a continuum.

There is an inverse relationship between the stresses and the strains if we subtract the hydrostatic stress from the stress tensor σ_{ij}. The resulting stress denoted by τ_{ij} is called the deviatoric stress,

$$\tau_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3, \tag{6}$$

where the unity tensor, Kronecker delta δ_{ij} is defined by

$$\delta_{ij} = \begin{matrix} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{matrix}.$$

Using (5a)–(6) in (3a)–(3c) and (4a)–(4c) we obtain the following deviatoric orthotropic stress–strain relation:

$$\tau_x = C^d_{11}\varepsilon_x + C^d_{12}\varepsilon_y + C^d_{13}\varepsilon_z \tag{7a}$$

$$\tau_y = C^d_{12}\varepsilon_x + C^d_{22}\varepsilon_y + C^d_{23}\varepsilon_z \tag{7b}$$

$$\tau_z = C^d_{13}\varepsilon_x + C^d_{23}\varepsilon_y + C^d_{33}\varepsilon_z. \tag{7c}$$

The deviatoric elastic constants in (7a)–(7c) are given by

$$C^d_{11} = (4E_x/E_y + 4E_x/E_z - 2)D \tag{8a}$$

$$C^d_{22} = (4E_x/E_z - 2E_x/E_y + 4)D \tag{8b}$$

$$C^d_{33} = (4E_x/E_y - 2E_x/E_z + 4)D \tag{8c}$$

$$C^d_{23} = (E_x/E_y + E_x/E_z - 5)D \tag{8d}$$

$$C^d_{13} = (E_x/E_z - 5E_x/E_y + 1)D \tag{8e}$$

$$C^d_{12} = (E_x/E_y - 5E_x/E_z + 1)D \tag{8f}$$

$$D = (2E_x/9)/[4E_x/E_y - (E_x/E_y - E_x/E_z + 1)^2]. \tag{8g}$$

The shear stresses τ_{yz}, τ_{xz} and τ_{xy} are unaffected and are still governed by (3d).

2.3. *Restrictions on the elastic constants*

The restrictions on the orthotropic elastic constants are given in Jones (1975). Applying the same requirements as in Jones (1975), we derive the following equivalent restrictions for the incompressible case:

$$D \geqslant 0 \tag{9a}$$

$$(1 + E_y/E_x - E_y/E_z)^2 < 4E_y/E_x \tag{9b}$$

$$(1 + E_z/E_x - E_z/E_y)^2 < 4E_z/E_x \tag{9c}$$

$$(1 + E_z/E_y - E_z/E_x)^2 < 4E_z/E_y. \tag{9d}$$

The four constraint equations in (9a)–(9d) are not all independent. It can be shown that the restrictions in (9a)–(9d) are satisfied by

$$(1 - \sqrt{E_x/E_y})^2 \leqslant E_x/E_z \leqslant (1 + \sqrt{E_x/E_y})^2. \tag{10}$$

The restriction in (10) may have important consequences for laminated fiber-reinforced composites, especially those made of high modulus fibers in nearly incompressible rubber-like matrices. Let us imagine that the x-axis is aligned perpendicular to the plane of the laminate. Then, the Young's modulus E_x of the resulting composite is essentially the same as that of the matrix, whereas the in-plane moduli E_y and E_z have high values because of the fibers. This structural arrangement can easily violate the restriction in (10) and thus restrict certain classes of deformations in the composite.

It should be noted that the above restriction in (10) may have some implications for incompressible plastic deformations of metals as well. When the effective plastic moduli do not satisfy the restriction in (10) volume preserving plastic flow may become restricted, which may lead to brittle fracture or damage.

2.4. *Arbitrary directions*

The tensorial representation theorems (Spencer, 1971) are used to derive the stress–strain relation in the global directions XYZ (Fig. 1). Mathematical representations for one tensor ε_{ij} and three unit vectors M_i, N_i and L_i are used by Boehler (1979) in order to derive transversely isotropic and orthotropic constitutive relations. We derive an equivalent orthotropic relation that has some advantage in representing the shear terms in comparison with Boehler (1979), using the same mathematical techniques [see Sutcu (1985)]. The accuracy of the presented expression can be checked easily, using coordinate transformations described in (2) and (3):

$$\sigma_{ij} = m_{pq}\varepsilon_{pq}(C_{11}m_{ij}+C_{12}n_{ij}+C_{13}l_{ij})+n_{pq}\varepsilon_{pq}(C_{12}m_{ij}+C_{22}n_{ij}+C_{23}l_{ij})$$
$$+l_{pq}\varepsilon_{pq}(C_{13}m_{ij}+C_{23}n_{ij}+C_{33}l_{ij})+2C_{44}(n_{ip}l_{jq}+n_{jp}l_{iq})\varepsilon_{pq}$$
$$+2C_{55}(m_{ip}l_{jq}+m_{jp}l_{iq})\varepsilon_{pq}+2C_{66}(m_{ip}n_{jq}+m_{jp}n_{iq})\varepsilon_{pq}, \quad (11)$$

where the tensors m_{ij}, n_{ij} and l_{ij} are calculated from

$$m_{pq} = M_pM_q \quad (12a)$$
$$n_{pq} = N_pN_q \quad (12b)$$
$$l_{pq} = L_pL_q. \quad (12c)$$

The quantities m_{pq}, n_{pq}, l_{pq} will be called geometric tensors. The elastic constants C_{ij} in (11) were defined in (4a)–(4c). The stresses σ_{ij} and the strains ε_{ij} in (11) are in the global directions XYZ (Fig. 1). The stresses σ'_{ij} and the strains ε'_{ij} in the preferred directions xyz can be obtained using the tensorial transformation expressions in (3). A fourth order tensorial representation related to (11) is defined in the appendix.

For incompressible materials, the deviatoric form of the stress–strain relation in (11) is obtained by replacing

$$\sigma_{ij} \quad \text{with} \quad \tau_{ij} \quad \text{and}$$
$$C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33} \quad \text{with} \quad C^d_{11}, C^d_{12}, C^d_{13}, C^d_{22}, C^d_{23}, C^d_{33}. \quad (13)$$

The inverse of (11) is

$$\varepsilon_{ij} = m_{pq}\sigma_{pq}(S_{11}m_{ij}+S_{12}n_{ij}+S_{13}l_{ij})+n_{pq}\sigma_{pq}(S_{12}m_{ij}+S_{22}n_{ij}+S_{23}l_{ij})$$
$$+l_{pq}\sigma_{pq}(S_{13}m_{ij}+S_{23}n_{ij}+S_{33}l_{ij})+(S_{44}/2)(n_{ip}l_{jq}+n_{jp}l_{iq})\sigma_{pq}$$
$$+(S_{55}/2)(m_{ip}l_{jq}+m_{jp}l_{iq})\sigma_{pq}+(S_{66}/2)(m_{ip}n_{jq}+m_{jp}n_{iq})\sigma_{pq}. \quad (14)$$

The elastic constants S_{11}, S_{12}, S_{13}, S_{22}, $S_{23}, \ldots, S_{66}$ in (14) are defined in terms of the engineering constants E_x, E_y, and so forth in Jones (1975) (see 2.23). The incompressible version of (14) is obtained by using the incompressible values for the Poisson's ratio ν_{zy}, ν_{zx}, ν_{yx} provided by (5a)–(5c).

2.5. *Orthotropic thermal expansion coefficients in arbitrary directions*

An orthotropic material may have three unequal thermal expansion coefficients α_x, α_y, α_z in the preferred directions, as discussed in Section 2.1. If these values are unequal, then the block D_1 in Fig. 1 may undergo shear deformation as a result of temperature change, because it is aligned in the off-axis directions of the material. The edges of the deformed block that were initially aligned along the X and Y axes now make an angle other than $\pi/2$ as a result of the temperature change. The difference between the new angle and $\pi/2$ for one degree temperature change, when expressed in radians, defines the shear thermal expansion coefficient $2\alpha_{12}$. Similarly, we can define $2\alpha_{13}$ and $2\alpha_{23}$ in the global XZ and YZ directions, respectively. Including the three normal thermal expansion coefficients α_{11}, α_{22} and α_{33} in the global X, Y and Z directions, respectively, there are six thermal expansion coefficients. The factor two is introduced in the shear terms because these six constants form the components of a second order tensor α_{ij}. Thus, the relationship between the six off-axis thermal expansions α_{11}, α_{22}, α_{33}, α_{23}, α_{13}, α_{12} and the three principal thermal expansion coefficients α_x, α_y, α_z is governed by the tensorial transformation law in (2)

$$\alpha_{ij} = \alpha_x M_i M_j + \alpha_y N_i N_j + \alpha_z L_i L_j, \tag{15}$$

or explicitly

$$\alpha_{11} = \alpha_x M_1^2 + \alpha_y N_1^2 + \alpha_z L_1^2 \tag{16a}$$

$$\alpha_{22} = \alpha_x M_2^2 + \alpha_y N_2^2 + \alpha_z L_2^2 \tag{16b}$$

$$\alpha_{33} = \alpha_x M_3^2 + \alpha_y N_3^2 + \alpha_z L_3^2 \tag{16c}$$

$$\alpha_{23} = \alpha_x M_2 M_3 + \alpha_y N_2 N_3 + \alpha_z L_2 L_3 \tag{16d}$$

$$\alpha_{13} = \alpha_x M_1 M_3 + \alpha_y N_1 N_3 + \alpha_z L_1 L_3 \tag{16e}$$

$$\alpha_{12} = \alpha_x M_1 M_2 + \alpha_y N_1 N_2 + \alpha_z L_1 L_2. \tag{16f}$$

The inverse relationship can be easily obtained using (3).

3. TRANSVERSE ISOTROPY

If at every point of a material there is one plane in which the properties are equal in all directions, then the material is termed transversely isotropic. A transversely isotropic material is characterized with only one preferred direction. The transversely isotropic stress–strain relationship in arbitrary directions is developed in a similar manner to the orthotropic case. We derive the stress–strain relation for region D_1 in Fig. 1 with built-in coordinate transformations. The material properties are obtained from testing region D_2, which is aligned along the preferred direction. The material is isotropic in the yz-plane in Fig. 1. Similar to the unit vectors $\bar{M}$, $\bar{N}$, $\bar{L}$ in (1a)–(1c), which are attached to the principal directions of orthotropy, we attach a single unit vector $\bar{M}$ along the axis of anisotropy in the x-direction. This unit vector is used to "remember" the preferred direction of the transverse isotropy.

The other two unit vectors $\bar{N}$ and $\bar{L}$ given by (1b) and (1c) can be chosen arbitrarily because these two directions lie in the plane of isotropy. For convenience, we choose one of the unit vectors $\bar{N}$ in the global YZ plane in Fig. 1. The third unit vector $\bar{L}$ is obtained by taking the vectorial cross product of $\bar{M}$ and $\bar{N}$:

$$\bar{N} = (M_3 j - M_2 k)/Q$$
$$\bar{L} = (-Q^2 i + M_1 M_2 j + M_1 M_3 k)/Q$$
$$Q = (M_2^2 + M_3^2)^{1/2}. \tag{17}$$

In the special case when the principal direction x is aligned in the global X direction, (i.e. $M_2 = M_3 = 0$), then the vectors $\bar{N}$ and $\bar{L}$ are given by $\bar{N} = j$, $\bar{L} = k$.

The stress–strain relationship given below is the linear part of the more general expression given by Boehler (1979):

$$\sigma_{ij} = \delta_{ij}(C_1 \varepsilon_{kk} + C_2 m_{pq}\varepsilon_{pq}) + m_{ij}(C_2 \varepsilon_{kk} + C_3 m_{pq}\varepsilon_{pq}) + C_4 \varepsilon_{ij} + C_5(m_{ip}\varepsilon_{pj} + m_{jp}\varepsilon_{pi}). \tag{18}$$

In comparison with the nine elastic constants of an orthotropic material, a transversely isotropic material possesses five constants. Incompressible transversely isotropic materials are characterized by three elastic constants.

The geometric tensor m_{ij} is constructed using the components of the preferred direction vector M_i as given by (12a). The stress σ_{ij} and the mechanical strain ε_{ij} in (18) are defined in the global coordinate system XYZ (Fig. 1). The stresses σ'_{ij} and the strains ε'_{ij} in the preferred directions can be obtained using the coordinate transformation defined in (3). The newly defined elastic constants C_1, C_2, C_3, C_4, C_5 in (18) are related to the more familiar engineering elastic constants as follows:

$$C_1 = k - G_T = C_{23} \tag{19a}$$

$$C_2 = (2\nu_A - 1)k + G_T = C_{12} - C_{23} \tag{19b}$$

$$C_3 = E_A + (2\nu_A - 1)^2 k - 4G_A + G_T = C_{11} + C_{22} - 2C_{12} - 4C_{66} \tag{19c}$$

$$C_4 = 2G_T = C_{22} - C_{23} \tag{19d}$$

$$C_5 = 2G_A - 2G_T = 2C_{66} - C_{22} + C_{23}. \tag{19e}$$

The definition of the elastic constants on the right hand side of eqns (19a)–(19e) is given in the Appendix.

The inverse of (18) is given by

$$\varepsilon_{ij} = \delta_{ij}(S_1\sigma_{kk} + S_2 m_{pq}\sigma_{pq}) + m_{ij}(S_2\sigma_{kk} + S_3 m_{pq}\sigma_{pq}) + S_4\sigma_{ij} + S_5(m_{ip}\sigma_{pj} + m_{jp}\sigma_{pi}), \tag{20}$$

where the newly defined elastic constants S_1, S_2, S_3, S_4, S_5 are given by

$$S_1 = -\nu_T/E_T = S_{23} \tag{21a}$$

$$S_2 = \nu_T/E_T - \nu_A/E_A = S_{12} - S_{23} \tag{21b}$$

$$S_3 = 1/E_A + 1/E_T + 2\nu_A/E_A - 1/G_A = S_{11} + S_{22} - 2S_{12} - S_{66} \tag{21c}$$

$$S_4 = (1 + \nu_T)/E_T = 1/(2G_T) = S_{22} - S_{23} \tag{21d}$$

$$S_5 = (1/G_A - 1/G_T)/2 = S_{66}/2 - S_{22} + S_{23}. \tag{21e}$$

The definition of the commonly used elastic constants S_{11}, S_{22}, and so forth is given in Christensen (1979).

The notation in (19) and (21) is chosen to correspond to

$$E_A = E_x, \quad E_T = E_y = E_z, \quad \nu_A = \nu_{xy} = \nu_{xz}$$
$$\nu_T = \nu_{yz} = \nu_{zy}, \quad \alpha_A = \alpha_x, \quad \alpha_T = \alpha_y = \alpha_z$$
$$G_A = G_{xy} = G_{xz}, \quad G_T = G_{yz}, \tag{22}$$

in comparison with the orthotropic case.

3.1. *Restrictions on the elastic constants and the incompressible Poisson's ratios*

The restrictions on the values that the elastic constants can take are obtained in a similar way to Jones (1975). These restrictions basically ensure that applied load produces deformation in the load direction, and the elastic stored energy in the material is positive for all possible deformations. Specializing the expressions in Jones (1975) for transverse isotropy, we obtain

$$E_A, E_T, G_A, G_T > 0 \tag{23a}$$

$$-1 < \nu_T < 1 \tag{23b}$$

$$\nu_A^2 \leqslant (1 - \nu_T)E_A/(2E_T). \tag{23c}$$

When the Poisson's ratios are governed by the following relations, the material behavior is incompressible

$$\nu_A = 1/2 \tag{24a}$$

$$\nu_T = 1 - E_T/(2E_A). \tag{24b}$$

By inserting the incompressible value of ν_T from (24b) into the lower limit in (23b), we arrive at the conclusion that incompressible deformations cannot occur in transversely isotropic materials where the axial direction x is softer than the transverse plane yz and the ratio of the moduli is such that the following condition is violated:

$$E_T < 4E_A. \tag{25}$$

This restriction is the transversely isotropic version of the orthotropic restriction given by (10).

3.2. *Incompressible stress–strain relationship*

The incompressible version of the transversely isotropic stress–strain law in (20) is obtained by calculating S_1, S_2, S_3, S_4, S_5 from (21a)–(21e) while replacing the Poisson's ratios ν_A and ν_T on the right-hand side with the incompressible values provided by (24a) and (24b). The resulting expression does not produce deformation under hydrostatic loading.

The incompressible version of the stress–strain law in (18) can only be defined between the deviatoric stress τ_{ij} defined in (6) and the strain ε_{ij}. The shear components of the deviatoric stress τ_{ij} are identical to the corresponding stress components σ_{ij}.

$$\tau_{ij} = \delta_{ij}(C_1^d\varepsilon_{kk} + C_2^d m_{pq}\varepsilon_{pq}) + m_{ij}(C_2^d\varepsilon_{kk} + C_3^d m_{pq}\varepsilon_{pq}) + C_4^d\varepsilon_{ij} + C_5^d(m_{ip}\varepsilon_{pj} + m_{jp}\varepsilon_{pi}). \tag{26}$$

The elastic constants C_i^d, $i = 1, 5$ are not all independent and are expressed in terms of the three independent elastic constants E_A, G_A and G_T (or E_T) as follows:

$$C_1^d = E_A/9 - G_T \tag{27a}$$

$$C_2^d = G_T - E_A/3 \tag{27b}$$

$$C_3^d = E_A + G_T - 4G_A \tag{27c}$$

$$C_4^d = 2G_T = 2E_T/(4 - E_T/E_A) \tag{27d}$$

$$C_5^d = 2G_A - 2G_T. \tag{27e}$$

The deviatoric relationship in (26) is similar to (18) except the constants C_i in (19a)–(19e) are replaced with C_i^d in (27a)–(27e). The bulk strain ε_{kk} in (26) is zero for incompressible deformations. This term is retained in order to maintain the symmetry of the fourth order tensor of elastic constants. The tensorial relationship in (26) is given in matrix form in the Appendix. It should be noted that the in-plane shear modulus G_T in (27d) becomes negative when the restriction in (25) is violated.

3.3. *Thermal expansion coefficients in arbitrary directions*

Similar to the orthotropic case discussed in Section 2.5, a transversely isotropic material may also deform in shear in the off-axis directions under temperature variation. The six off-axis thermal expansion coefficients α_{11}, α_{22}, α_{33}, α_{23}, α_{13}, α_{12} are related to the two principal thermal expansion coefficients α_A and α_T (see (22)),

$$\alpha_{ij} = \alpha_T\delta_{ij} + (\alpha_A - \alpha_T)M_iM_j \tag{28}$$

or explicitly

$$\alpha_{11} = \alpha_T + M_1^2(\alpha_A - \alpha_T) \tag{29a}$$

$$\alpha_{22} = \alpha_T + M_2^2(\alpha_A - \alpha_T) \tag{29b}$$

$$\alpha_{33} = \alpha_T + M_3^2(\alpha_A - \alpha_T) \tag{29c}$$

$$\alpha_{23} = M_2M_3(\alpha_A - \alpha_T) \tag{29d}$$

$$\alpha_{13} = M_1M_3(\alpha_A - \alpha_T) \tag{29e}$$

$$\alpha_{12} = M_1 M_2(\alpha_A - \alpha_T). \quad (29f)$$

The subscripts 1, 2, 3 in (29a)–(29f) indicate the global coordinate axes X, Y, Z respectively, defined in the off-axis directions (Fig. 1).

4. DETERMINATION OF THE PREFERRED MATERIAL DIRECTIONS

Conceptual or actual experiments can be devised in order to determine the material symmetry planes. The unknowns are the direction cosines of the material-fixed unit vectors M_i, N_i and L_i which were used in the previous sections in order to "remember" the preferred directions. Perhaps, the easiest test is to subject the material with unknown properties to a temperature change. Shear distortion upon uniform heating is a good indication that the material is anisotropic and that the test specimen is aligned in the off-axes directions. Shear deformations under uniaxial or biaxial normal loading can be interpreted in a similar manner. As discussed previously, applying hydrostatic pressure offers a convenient method for determining the material symmetry planes, if the material is orthotropic.

4.1. *Temperature change in an orthotropic material*

The orthotropic block D_1 in Fig. 1, upon heating, undergoes length changes along its edges and angle changes between the edges, so that the corners are no longer at right angles. The percentage length changes along the three mutually orthogonal edges are given by the normal strain components ε_{11}, ε_{22} and ε_{33}. The angle changes, when expressed in radians and divided by 2, are represented by the shear strain components ε_{23}, ε_{13} and ε_{12}. The six strain components are divided by the temperature variation ΔT, in order to arrive at the six thermal expansion coefficients, α_{11}, α_{22}, α_{33}, α_{23}, α_{13}, α_{12}.

The relationship between the six off-axes thermal expansion coefficients α_{ij} and the three preferred thermal expansion coefficients α_x, α_y and α_z is given in (15)–(16f). In mathematical terms, the determination of the preferred directions is an eigenvalue problem, which can be demonstrated easily by taking the inner product of α_{ij} in (15) with the unit vector M_i,

$$\alpha_{ij} M_j = \alpha_x M_i \quad (30a)$$

$$\alpha_{ij} N_j = \alpha_y N_i \quad (30b)$$

$$\alpha_{ij} L_j = \alpha_z L_i, \quad (30c)$$

using the orthogonality conditions on the unit vectors. Thus, the three preferred thermal expansion coefficients α_x, α_y, α_z are the eigenvalues of the thermal expansion tensor α_{ij}, and the nine direction cosines M_1, M_2, M_3, N_1, N_2, N_3, L_1, L_2, L_3 are the direction cosines of the three mutually orthogonal eigenvectors of α_{ij}. If two of the eigenvalues are equal to one another, then the material is transversely isotropic as far as the thermal expansion behavior is considered. The actual mechanical anisotropy may be more complicated. The method of hydrostatic pressure provides a more reliable method and is discussed next.

4.2. *Hydrostatic pressure*

If the material is orthotropic and compressible, then the preferred directions of the material coincide with the eigenvectors of the strain tensor ε_{ij}, which is produced under hydrostatic pressure, $\sigma_{ij} = p\delta_{ij}$. The term p denotes the hydrostatic pressure, and δ_{ij} is the Kronecker delta defined after (6). The strain response ε_{ij} is calculated from (14),

$$\varepsilon_{ij}/p = M_i M_j (S_{11}+S_{12}+S_{13}) + N_i N_j (S_{22}+S_{12}+S_{23}) + L_i L_j (S_{33}+S_{13}+S_{23}). \quad (31)$$

Furthermore, the eigenvalues of the strain tensor ε_{ij} provide some information about the elastic constants. The inner product of the strain tensor ε_{ij} in (31) with the unit vectors M_i, N_i and L_i produces the eigenvalues in terms of the elastic constants:

$$\varepsilon_{ij}M_j = M_i(S_{11}+S_{12}+S_{13})p = \lambda_1 M_i \quad (32a)$$

$$\varepsilon_{ij}N_j = N_i(S_{22}+S_{12}+S_{23})p = \lambda_2 N_i \quad (32b)$$

$$\varepsilon_{ij}L_j = L_i(S_{33}+S_{13}+S_{23})p = \lambda_3 L_i. \quad (32c)$$

The eigenvalues are denoted by λ_1, λ_2 and λ_3. If two of the eigenvalues are equal then the material may be transversely isotropic. Note that the shear moduli do not appear in (32a)–(32c). It is also worthwhile to mention that a mathematical analogy exists between the method of hydrostatic pressure and the method of uniform heating. This analogy can be seen by comparing the eigenvalues predicted in (30a)–(30c) and (32a)–(32c).

Determining the eigenvectors of the strain tensor under hydrostatic pressure, conveniently filters out most of the unknown elastic constants so that the preferred directions are derived easily. Alternatively, one can use direct algebraic methods by considering six independent equations that are derived from (14) under hydrostatic pressure. The algebraic solutions for orthotropy are quite lengthy because of the non-linear coupling of the unknown terms. However, the particular equations that are derived for transverse isotropy from (20) are manageable.

Let us assume that six strain components have been measured under hydrostatic pressure in a transversely isotropic material and we wish to determine the three direction cosines M_1, M_2 and M_3 of the preferred direction from this information. Setting $\sigma_{ij} = p\delta_{ij}$ in the transversely isotropic stress–strain law in (20) shows that there are five unknowns—the three direction cosines M_1, M_2, M_3 and two combination elastic constants $(3S_1+S_2+S_4)$ and $(3S_2+S_3+2S_5)$. Six components of the strains in (20) and the unit magnitude condition on the unit vector $\vec{M}$ provide seven equations for the five unknowns. Thus, two of the measured strain components under hydrostatic loading are dependent on the other measured strain components, if the material is truly transversely isotropic:

$$\varepsilon_{13} = \varepsilon_{12}\varepsilon_{23}(\varepsilon_{11}-\varepsilon_{33})/(\varepsilon_{12}^2-\varepsilon_{23}^2) \quad (33a)$$

$$\varepsilon_{22} = \varepsilon_{11} + \frac{\varepsilon_{12}^2(\varepsilon_{33}-\varepsilon_{11})^2-(\varepsilon_{12}^2-\varepsilon_{23}^2)^2}{(\varepsilon_{33}-\varepsilon_{11})(\varepsilon_{12}^2-\varepsilon_{23}^2)}. \quad (33b)$$

With these constraints, the number of equations is reduced to the number of the unknowns. Equations (33a) and (33b) make it possible to check for transverse isotropy under hydrostatic pressure without the need to solve an eigenvalue problem.

Let us assume that we wish to determine an approximate plane of transverse isotropy for a material that does not satisfy (33a) and (33b). Since the actual material behavior is not transversely isotropic, the two equations in (33) represent the error in the approximation. We must ensure that the stiffest and the softest directions are included in the analysis. We may underestimate the real anisotropy if we ignore either the stiffest or the softest material directions and consider the intermediate or "in-between" stiff directions. For this purpose, it is advantageous to choose the global XYZ coordinate system such that the X and Z axes are aligned in the stiff and in the soft directions of the material respectively. The stiff and the soft directions of a material can be determined by applying hydrostatic pressure and measuring strains. The maximum and minimum normal strains correspond to the soft and the stiff directions respectively.

The global coordinate system XYZ in Fig. 1 is chosen so that the normal strain components are ordered in the following manner: $|\varepsilon_{11}| < |\varepsilon_{22}| < |\varepsilon_{33}|$ under hydrostatic pressure. This choice ensures that the stiff material direction lies close to the X direction, whereas the more compliant direction is close to the Z direction. When the stiff direction lies in the plane of isotropy, the shear strain components are ordered such that $|\varepsilon_{23}| > |\varepsilon_{13}| > |\varepsilon_{12}|$ under hydrostatic loading. On the other hand, if the out-of-plane direction is stiffer, then the shear strains are governed by $|\varepsilon_{23}| < |\varepsilon_{13}| < |\varepsilon_{12}|$, provided that the material is indeed transversely isotropic. If the shear strains under hydrostatic loading are not ordered as such, then the material has a more complicated symmetry class than transverse isotropy. In order to emphasize the stiff and the soft directions and ignore the intermediary directions, the two extra equations that contain ε_{22} and ε_{13} on the left hand side in (20) will be dropped in the approximation.

4.3. *Uniaxial loading*

We introduce the method of the uniaxial loading primarily for materials that do not deform under hydrostatic stress. The method will be discussed for compressible materials, and subsequently will be specialized for incompressible materials. Our purpose is to determine the approximate symmetry planes in a given material so that the stress–strain response is approximated by orthotropy or transverse isotropy. The global coordinate system XYZ in Fig. 1 is chosen such that the stiff material direction lies close to the X direction, whereas the compliant direction is close to the Z direction. This choice is accomplished by applying unit uniaxial stress in three mutually orthogonal directions and subsequently labeling the directions with X, Y, Z such that the normal strain components in the applied load directions are ordered in the following manner: $|\varepsilon_{11}| < |\varepsilon_{22}| < |\varepsilon_{33}|$.

4.3.1. *Orthotropy*. Three uniaxial loadings are needed to completely characterize the state of orthotropy in a material. These three uniaxial loadings can be in arbitrary directions; however, they cannot lie in a single plane. The uniaxial loadings in the proposed method are always chosen in the global X, Y and Z directions. From these three tests, 15 independent strain components are measured. Using these measured values, 15 equations are generated from the orthotropic stress–strain relation in (14) for solving the following 18 unknowns:

$$M_1, M_2, M_3, N_1, N_2, N_3, L_1, L_2, L_3,$$
$$S_{11}, S_{22}, S_{33}, S_{23}, S_{13}, S_{12}, S_{44}, S_{55}, S_{66}. \tag{34}$$

Additional six equations are provided by the conditions on the unit vectors $\bar{\text{M}}$, $\bar{\text{N}}$ and $\bar{\text{L}}$ and given below:

$$\begin{aligned} M_1^2+M_2^2+M_3^2 = N_1^2+N_2^2+N_3^2 = L_1^2+L_2^2+L_3^2 &= 1 \\ M_1N_1+M_2N_2+M_3N_3 &= 0 \\ M_1L_1+M_2L_2+M_3L_3 &= 0 \\ N_1L_1+N_2L_2+N_3L_3 &= 0. \end{aligned} \tag{35}$$

Thus, the 18 unknowns in (34) are overly constrained by 21 equations that result from three uniaxial loadings. If the material behavior is orthotropic, then the extra three equations are identically satisfied; otherwise, they represent error. In order to ensure that the error occurs in the intermediary direction, rather than in the stiff or the soft directions, we choose the following three equations to represent the error; the equations that correspond to (i) the normal strain ε_{22} in (14) when the uniaxial load is applied in the direction Y, (ii) the shear strain ε_{13} in (14) when the uniaxial load is applied in the direction X and (iii) the shear strain ε_{13} in (14) when the uniaxial load is applied in the direction Z. Dropping these three equations, the number of unknowns matches the number of equations, and a solution is possible.

It is possible that the material symmetry class is simpler than orthotropy. In this case the solution of the above equations is not unique and the extra equations that represent error are satisfied identically. If this is the case, then one should investigate the possibility of transverse isotropy.

The above-mentioned method is applicable to incompressible materials as well. Incompressibility can be detected by summing the three measured normal strain components from each uniaxial test. If all three summations produce nearly zero, then the material is incompressible. Since the normal strain components are related, the three uniaxial tests produce 12 rather than 15 independent equations for incompressible materials. The number of unknown elastic constants is six rather than nine. Thus, we again have three extra equations that represent error.

4.3.2. *Transverse isotropy*. Two uniaxial loadings are needed to completely characterize a transversely isotropic material. They can be in arbitrary directions so long as they do not coincide. The uniaxial loadings in this paper are always chosen in the stiff and in the soft X and Z directions. From these two tests, 11 independent strain components are measured. Using these measured values, 11 equations are generated from the transversely isotropic stress–strain relation in (20)–(21e) for solving the following seven unknowns:

$$E_A, E_T, G_A, G_T, \nu_A, M_1, M_2. \tag{36}$$

The third component M_3 of the unit vector $\bar{M}$ is related to M_1 and M_2 in (35). Unlike the hydrostatic loading, all five elastic constants are included in the stress–strain relation (20). The seven unknowns in (36) are overly constrained by 11 equations. If the material behavior is truly transversely isotropic, then the extra four equations are identically satisfied; otherwise, they represent error. In order to ensure that the error occurs in the intermediary direction, rather than in the stiff or the soft directions, we disregard the four equations that contain ε_{22} and ε_{13} on the left hand side in (20). Dropping these four equations, the number of seven unknowns matches the number of seven equations, and a solution is possible.

If the solution to the equations in (20) is not unique, then the material is isotropic. The above-mentioned method is applicable to incompressible materials as well. For incompressible materials, two uniaxial tests produce nine rather than 11 independent equations, and the number of unknown elastic constants is three rather than five. Thus, we again have four extra equations that represent error.

5. DISCUSSION AND SUMMARY

Elastic stress–strain relations in arbitrary material directions are presented for transversely isotropic and orthotropic materials. The relations given by (11), (14), (15), (18) and (20) are expressed in terms of the elastic constants that are measured along the preferred material directions and three unit vectors $\bar{M}$, $\bar{N}$ and $\bar{L}$ in order to "remember" the preferred directions. The need for rotating the stress and the strain tensors into the preferred directions is eliminated because the preferred directions are built into the stress–strain relations. In the case of transverse isotropy we only need to "remember" the direction perpendicular to the plane of isotropy.

Incorporating material-attached unit vectors into the stress–strain relationship makes it possible to use tensor algebra for determining the symmetry properties from the mechanical response of the material under load. The unknowns are simply the direction cosines of the unit material vectors. We can than systematically reduce the 21 elastic constants to simpler symmetry subclasses shown on pages 296–301 in Nye (1979). A computational algorithm is needed for this purpose; however, this will be done elsewhere.

The special case when the material is incompressible is treated by providing deviatoric orthotropic and transversely isotropic stress–strain relations (see (7a)–(8g), (13) and (26)–(27e)). The deviatoric stress excludes the hydrostatic pressure in the material. The strains are such that the material volume remains unchanged during deformation. The deviatoric constitutive relationships are useful for certain composite materials which are treated as incompressible matrices reinforced with inextensible cords [see Mulhern *et al.* (1967); Everstine and Pipkin (1973); Spencer (1974)]. It should be noted that an incompressible material can have a maximum of 15 anisotropic elastic constants in comparison with the 21 elastic constants of a general anisotropic body.

The restrictions on the orthotropic elastic constants in Jones (1975) have been extended to incompressible deformations and to the case of transverse isotropy. When the elastic constants violate these restrictions, an inhomogeneous material cannot effectively deform as an equivalent homogeneous medium under all loading situations. Some deformation modes will be constrained, and large stress concentrations may result at the interface of the constituents when a restricted class of deformation is activated by the applied load. This will cause localized deformation, such as plastic flow, damage or fracture, in order to accommodate the applied load. An example of this phenomena, is the common occurrence of interlaminar shear failure in laminated composites. The restrictions given by (10) and (25) for incompressible materials become important for laminated composite structures where the out-of-plane effective elastic modulus is smaller than the in-plane stiffness. Such will be the case when the fibers are much stiffer than the matrix. The general restrictions given by (23b)–(23c) for transverse isotropy, and for orthotropy by Jones (1975), are probably commonly violated for these laminated composites. It can be shown that the restricted class of deformations for laminated composites include the in-plane shear and the out-of-plane normal loading. Perhaps, one reason for the success of the laminated composites is the fact that the edges of the composite are usually constrained in such a manner as to produce even strain distribution in all layers and thus preventing edge delamination and interlaminar shear failure. Another reason may be the ability of the matrix to deform non-linearly.

Similar arguments can be made about the elastic energy content of the body. When the restrictions on the elastic constants are violated, the structure cannot contain elastic energy as an equivalent homogeneous medium. All of the energy input into the material is consumed by increasing the stress concentrations at singular locations rather than by volume deformation in the body.

Laminated structures using brittle fiber and matrix combinations may present interesting challenges in terms of preventing interlaminar shear failure. Let us assume that enough number symmetric layups are stacked so that the in-plane response is nearly isotropic. Equation (25) indicates that edge delamination or interlaminar shear failure will be a problem if the composite is nearly incompressible and the matrix and the fiber Young's moduli are such that $E_T > 4E_X$. Using Christensen's effective shear modulus for the composite, it can be shown that $E_T > 4E_X$ when the fiber and matrix Young's moduli are governed by $E_f > 33.4E_m$, $E_f > 26.3E_m$ or $E_f > 17.9E_m$ for the matrix Poisson's ratios of 0.49, 0.45 and 0 respectively. The fiber volume fraction is taken to be 0.45. The above values are calculated by assuming perfect interface bonding and fiber spacing. Processing flaws and uneven fiber spacing may reduce the out-of-plane modulus E_X considerably lower than the calculated values. Experimental measurements of the out-of-plane modulus for laminated composites is needed. For woven fiber composites, we suggest adding fibers in the soft composite direction until the effective Young's moduli of the resulting composite satisfies the orthotropic restriction in (10), although this relationship was derived for incompressible deformations.

Uniform heating (without microstructural change) and hydrostatic loading provide considerable information about the symmetry planes of a material. In both cases, the eigenvectors of the resulting strain tensor directly coincide with the preferred directions of the material. The method of hydrostatic stress is applicable only if the material is compressible. For incompressible materials the method of uniform heating can be used; however, we can determine only thermal expansion anisotropy from this method. Thermal expansion anisotropy is generally a simpler symmetry class than the actual mechanical anisotropy. The direction cosines M_i, N_i and L_i are treated as unknowns for the determination of the approximate symmetry planes in the material. When the material is approximated as an orthotropic medium, three preferred directions are calculated, whereas only one preferred direction is calculated in order to approximate the material behavior as transversely isotropic.

Approximate material symmetry planes can also be determined from uniaxial loadings. Three uniaxial loadings are needed for characterizing the state of orthotropy, whereas two are needed for transverse isotropy. Approximations are carried out so as to include the stiffest and the softest material directions.

The stiffest material direction in uniaxial loading may not coincide with the stiffest direction under hydrostatic pressure for some classes of laminated or woven fiber composites and in some single crystal materials. In structural applications, it is desirable to choose the main load bearing direction in the stiff direction obtained under hydrostatic loading in order to minimize shear distortions. Thermal shear distortions are also minimized with the above choice of the loading directions, because the principal strain directions under hydrostatic loading coincide with the principal thermal strain directions.

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APPENDIX

A1. *Representation of the elastic fourth order tensor for orthotropic materials*

The orthotropic elastic constitutive relationship in (11) can be expressed in terms of a fourth order tensor,

$$\sigma_{ij} = C_{ijpq}\varepsilon_{pq}. \tag{A1}$$

The fourth order elastic tensor C_{ijpq} is expressed in terms of the orthotropic elastic constants C_{11}, C_{22}, and so forth and the geometric tensors m_{ij}, n_{ij} and l_{ij} that are defined in (12a)–(12c) as follows:

$$C_{ijpq} = m_{pq}(C_{11}m_{ij}+C_{12}n_{ij}+C_{13}l_{ij})+n_{pq}(C_{12}m_{ij}+C_{22}n_{ij}+C_{23}l_{ij})+l_{pq}(C_{13}m_{ij}+C_{23}n_{ij}+C_{33}l_{ij})$$
$$+2C_{44}(n_{ip}l_{jq}+n_{jp}l_{iq})+2C_{55}(m_{ip}l_{jq}+m_{jp}l_{iq})+2C_{66}(m_{ip}n_{jq}+m_{jp}n_{iq}). \tag{A2}$$

A similar fourth order tensor S_{ijpq} can be defined for the inverse relationship (14). The factor 2 in the last three terms in (A2) is replaced by (1/2) for this purpose.

A2. *Transversely isotropic stress–strain relation in the preferred direction*

The transversely isotropic constitutive relationship is commonly given in the preferred directions of the material [see, for example, Christensen (1979)]. When the preferred direction vector $\tilde{M}$ is aligned along the global X axis, i.e. $M_1 = 1$, $M_2 = M_3 = 0$, the general relationship given in (18) reduces to

$$\sigma_{xx} = C_{11}\varepsilon_x + C_{12}(\varepsilon_y+\varepsilon_z) \tag{A3a}$$
$$\sigma_{yy} = C_{12}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z \tag{A3b}$$
$$\sigma_{zz} = C_{12}\varepsilon_x + C_{23}\varepsilon_y + C_{22}\varepsilon_z \tag{A3c}$$
$$\sigma_{yz} = (C_{22}-C_{23})\varepsilon_{yz} = 2G_T\varepsilon_{yz} \tag{A3d}$$
$$\sigma_{xz} = 2C_{66}\varepsilon_{xz} \tag{A3e}$$
$$\sigma_{xy} = 2C_{66}\varepsilon_{xy}. \tag{A3f}$$

The five elastic constants C_{11}, etc., (Christensen, 1979) are related to the engineering constants [see Lekhnitskii (1981); Hashin (1979)]:

$$C_{11} = E+4kv_A^2,\quad C_{12} = 2v_A k$$
$$C_{22} = k+G_T,\quad C_{23} = k-G_T,\quad C_{66} = G_A. \tag{A4}$$

The transverse bulk modulus k is given by [see Hashin (1979)]:

$$2k = E_T/(1-\nu_T-2\nu_A^2 E_T/E_A).$$

A3. *The matrix form of the incompressible transversely isotropic relationship in* (26)

The stress–strain relations are commonly expressed in matrix form [see Jones (1975) or Christensen (1979)]. For this purpose the components of stress and strain are stored in 6×1 vectors as follows:

$$\tau_1 = \tau_{11},\quad \tau_2 = \tau_{22},\quad \tau_3 = \tau_{33},\quad \tau_4 = \tau_{23},\quad \tau_5 = \tau_{13},\quad \tau_6 = \tau_{12}$$
$$\varepsilon_1 = \varepsilon_{11},\quad \varepsilon_2 = \varepsilon_{22},\quad \varepsilon_3 = \varepsilon_{33},\quad \varepsilon_4 = 2\varepsilon_{23},\quad \varepsilon_5 = 2\varepsilon_{13},\quad \varepsilon_6 = 2\varepsilon_{12}. \tag{A5}$$

The matrix form for (26) is given in the preferred directions only. For this purpose, we set $m_{11} = 1$, and all other $m_{ij} = 0$ in (26). Using the notation in (A5),

$$\tau_i = C^d_{ij}\varepsilon_j,\quad i,j = 1,6, \tag{A6}$$

where summation over repeated indices is implied. The components of the elastic matrix C^d_{ij} are given by

$$C^d_{11} = 4E_A/9 \tag{A7a}$$
$$C^d_{12} = C^d_{13} = -2E_A/9 \tag{A7b}$$
$$C^d_{22} = C^d_{33} = E_A/9+G_T \tag{A7c}$$
$$C^d_{23} = E_A/9-G_T. \tag{A7d}$$

The explicit form of the deviatoric stress–strain relation in (A6) is the following:

$$\tau_{11} = C^d_{11}\varepsilon_{11}+C^d_{12}(\varepsilon_{22}+\varepsilon_{33})$$
$$\tau_{22} = C^d_{12}\varepsilon_{11}+C^d_{22}\varepsilon_{22}+C^d_{23}\varepsilon_{33}$$
$$\tau_{33} = C^d_{12}\varepsilon_{11}+C^d_{23}\varepsilon_{22}+C^d_{22}\varepsilon_{33}$$
$$\tau_{23} = 2G_T\varepsilon_{23},\quad \tau_{13} = 2G_A\varepsilon_{13},\quad \tau_{12} = 2G_A\varepsilon_{12}. \tag{A8}$$